Teorema di Gagliardo

In questa nota useremo la notazione

$$\mathbb{R}^d_+ := \Big\{x = (x', x_d) \in \mathbb{R}^d \ : \ x' \in \mathbb{R}^{d-1},\ x_d > 0\Big\}.$$

1. Disuguaglianza della traccia in $\mathbb{R}^d_+$

Lemma 1. *Sia $\varphi \in C(\overline{\mathbb{R}}^d_+) \cap C^1_c(\mathbb{R}^d_+)$ una funzione definita su $\mathbb{R}^d_+$. Allora,*

$$\int_{\mathbb{R}^{d-1}} \varphi(x', 0)\, dx' \le \int_{\mathbb{R}^d_+} |\nabla \varphi|\, dx.$$

Proof. Sia $x' \in \mathbb{R}^{d-1}$. Allora, per ogni $r \in (0, 1)$, abbiamo

$$\varphi(x', 0) \le \int_0^{+\infty} |\partial_d \varphi(x', x_d)|\, dx_d \le \int_0^{+\infty} |\nabla \varphi|(x', x_d)\, dx_d$$

Integrando in x', abbiamo la tesi. □

Lemma 2. *Sia $\varphi \in C(\overline{\mathbb{R}}^d_+) \cap C^1_c(\mathbb{R}^d_+)$ una funzione definita su $\mathbb{R}^d_+$. Allora,*

$$\int_{\mathbb{R}^{d-1}} \varphi^2(x', 0)\, dx' \le \int_{\mathbb{R}^d_+} \varphi^2\, dx + \int_{\mathbb{R}^d_+} |\nabla \varphi|^2\, dx.$$

Teorema 3 (Teorema della traccia)**.** *Sia $u \in H^1(\mathbb{R}^d_+)$. Allora, $u \in L^2(\mathbb{R}^{d-1})$ e*

$$\int_{\mathbb{R}^{d-1}} u^2(x', 0)\, dx' \le \int_{\mathbb{R}^d_+} u^2\, dx + \int_{\mathbb{R}^d_+} |\nabla u|^2\, dx.$$

2. Diseguaglianza integrale di Minkowski

Teorema 4. *Siano $X \subset \mathbb{R}^m$, $Y \subset \mathbb{R}^n$ due insiemi di misura finita e*

$$F : X \times Y \to \mathbb{R}$$

una funzione, misurabile, limitata e positiva. Sia $p > 1$. Allora

$$\left(\int_X \left(\int_Y F(x,y)\, dy\right)^p dx\right)^{1/p} \le \int_Y \left(\int_X F(x,y)^p\, dx\right)^{1/p} dy\,.$$

Proof. Definiamo

$$u(x) = \int_Y F(x, y)\, dy.$$

Allora,

$$\begin{aligned}
\int_X \left(\int_Y F(x,y)\, dy\right)^p dx &\le \int_X u(x)^{p-1} \int_Y F(x,y)\, dy\, dx \\
&= \int_X \int_Y u(x)^{p-1} F(x,y)\, dy\, dx \\
&= \int_Y \int_X u(x)^{p-1} F(x,y)\, dx\, dy \\
&\le \int_Y \left(\int_X u(x)^p\, dx\right)^{\frac{p-1}{p}} \left(\int_X F(x,y)^p\, dx\right)^{\frac{1}{p}} dy \\
&= \left(\int_X u(x)^p\, dx\right)^{\frac{p-1}{p}} \int_Y \left(\int_X F(x,y)^p\, dx\right)^{\frac{1}{p}} dy,
\end{aligned}$$

il che conclude la dimostrazione. □

3. Disuguaglianza di Hardy integrale

Teorema 5. *Sia $f : \mathbb{R}_+ \to \mathbb{R}_+$ una funzione misurabile, positiva ed a supporto compatto in $[0, +\infty)$. Sia $p > 1$. Allora,*

$$\int_0^{+\infty} \left(\frac{1}{x} \int_0^x f(s)\, ds \right)^p dx \le \left(\frac{p}{p-1} \right)^p \int_0^{+\infty} f(x)^p\, dx.$$

Proof.

$$\int_0^{+\infty} \left(\frac{1}{x} \int_0^x f(s)\, ds \right)^p dx = \int_0^{+\infty} \left(\int_0^1 f(tx)\, dt \right)^p dx$$

Quindi, per la disuguaglianza integrale di Minkowski,

$$\begin{aligned} \left(\int_0^{+\infty} \left(\frac{1}{x} \int_0^x f(s)\, ds \right)^p dx \right)^{1/p} &= \left(\int_0^{+\infty} \left(\int_0^1 f(tx)\, dt \right)^p dx \right)^{1/p} \\ &\le \int_0^1 \left(\int_0^{+\infty} f(tx)^p\, dt \right)^{1/p} dx \\ &= \int_0^1 \left(\frac{1}{x} \int_0^{+\infty} f(s)^p\, ds \right)^{1/p} dx \\ &= \frac{p}{p-1} \left(\int_0^{+\infty} f(s)^p\, ds \right)^{1/p}. \end{aligned}$$

□

4. Teorema di Gagliardo

Teorema 6. *Sia $u \in H^1(\mathbb{R}^d_+)$. Definiamo la funzione $v \in L^2(\mathbb{R}^{d-1})$ come*

$$v(x') = u(x', x_d).$$

Allora,

$$\int_{\mathbb{R}^{d-1}} \int_{\mathbb{R}^{d-1}} \frac{|v(x') - v(y')|^2}{|x' - y'|^d}\, dx' dy' \le 8 d \omega_d \int_{\mathbb{R}^d_+} |\nabla u|^2\, dx,$$

dove ω_d è il volume della palla unitaria in $\mathbb{R}^d$.

Proof. Osserviamo che

$$\int_{\mathbb{R}^{d-1}} \int_{\mathbb{R}^{d-1}} \frac{|v(y') - v(x')|^2}{|y' - x'|^d}\, dx' dy' = \int_{\mathbb{R}^{d-1}} \int_{\mathbb{R}^{d-1}} \frac{|v(x' + 2h') - v(x')|^2}{|h'|^d}\, dx' dh'.$$

Inoltre, per la disuguaglianza triangolare, abbiamo

$$|v(x' + 2h') - v(x')| \le |u(x' + 2h', 0) - u(x' + h', |h'|)| + |u(x' + h', |h'|) - u(x', 0)|.$$

Di conseguenza,

$$\begin{aligned} \int_{\mathbb{R}^{d-1}} \int_{\mathbb{R}^{d-1}} \frac{|v(x' + 2h') - v(x')|^2}{|h'|^d}\, dx' dh' &= \int_{\mathbb{R}^{d-1}} \int_{\mathbb{R}^{d-1}} \frac{\left| u(x' + 2h', 0) - u(x' + h', |h'|) \right|^2}{|h'|^d}\, dx' dh' \\ &\quad + \int_{\mathbb{R}^{d-1}} \int_{\mathbb{R}^{d-1}} \frac{\left| u(x' + h', |h'|) - u(x', 0) \right|^2}{|h'|^d}\, dx' dh' \\ &= 2 \int_{\mathbb{R}^{d-1}} \int_{\mathbb{R}^{d-1}} \frac{\left| u(x' + h', |h'|) - u(x', 0) \right|^2}{|h'|^d}\, dh' dx'. \end{aligned}$$

In coordinate polari, abbiamo

$$\int_{\mathbb{R}^{d-1}} \frac{\left| u(x' + h', |h'|) - u(x', 0) \right|^2}{|h'|^d}\, dh' = \int_{\mathbb{S}^{d-2}} \int_0^{+\infty} \frac{\left| u(x' + r\theta', r) - u(x', 0) \right|^2}{r^2}\, dr\, d\theta'$$

Osserviamo che

$$|u(x' + r\theta', r) - u(x', 0)| = \left| \int_0^r \partial_t \big[u(x' + \theta' t, t) \big]\, dt \right| \le \int_0^r |\nabla u|(x' + \theta' t, t)\, dt.$$

Di conseguenza,

$$\int_{\mathbb{R}^{d-1}}\int_{\mathbb{R}^{d-1}} \frac{|u(x'+h',|h'|)-u(x',0)|^2}{|h'|^d}\,dh'dx' = \int_{\mathbb{R}^{d-1}} dx' \int_{\mathbb{S}^{d-2}} \int_0^{+\infty} \frac{1}{r^2}\left|\int_0^r |\nabla u|(x'+\theta' t,t)\,dt\right|^2 dr\,d\theta'$$

Usando la disuguaglianza di Hardy, otteniamo

$$\int_0^{+\infty}\left|\frac{1}{r}\int_0^r |\nabla u|(x'+\theta' t,t)\,dt\right|^2 dr \le 4\int_0^{+\infty} |\nabla u|^2(x'+\theta' t,t)\,dt\,.$$

Quindi,

$$\begin{aligned}\int_{\mathbb{R}^{d-1}}\int_{\mathbb{R}^{d-1}} \frac{|u(x'+h',|h'|)-u(x',0)|^2}{|h'|^d}\,dh'dx' &\le 4\int_{\mathbb{R}^{d-1}} dx' \int_{\mathbb{S}^{d-2}} \int_0^{+\infty} |\nabla u|^2(x'+\theta' t,t)\,dt\,d\theta' \\ &\le 4d\omega_d \int_{\mathbb{R}^d_+} |\nabla u|^2\,dx.\end{aligned}$$

□

Corollario 7. *Sia B_1 la palla unitaria in $\mathbb{R}^d$. Allora, esiste una costante dimensionale $C_d > 0$ tale che per ogni $u \in H^1(B_1)$*

$$\int_{\partial B_1}\int_{\partial B_1} \frac{|u(y')-u(x')|^2}{|y'-x'|^d}\,dx'\,dy' \le C_d \|\nabla u\|^2_{L^2(B_1)}.$$